Robust Optimization and Its Tractability Under Uncertainty Sets

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1. Abstract

Robust Optimization(RO) is an emerging field in the recent days to handle uncertainty associated with optimization problems. The main focus of Robust Optimization is to solve optimization problems due to uncertainty and subject to uncertainty affected constraints where the constraints must be satisfied for all realizations of uncertain values within a given uncertainty set like boxes, polyhedral, ellipsoids. The challenge of RO is to reformulate the constraints so that the uncertain optimization problem is transformed into a tractable deterministic form with the help of robust counterpart.

2. Introduction

RO Problem: Robust optimal design is one with the best worst-case performances. The general formulation of an uncertain linear optimization problem is,

$$\min_x \,[c^T x : Ax \le b]_{(c,A,b)} \in \mathcal{U}$$

In more precise form it looks,

$$\min_x \; c^T x \quad \text{s.t} \quad Ax \le b. \qquad (1)$$

where $c \in R^n$, $A \in R^{m\times n}$, $b \in R^m$ denote the uncertain set and $\mathcal{U}$ denotes the user specified uncertain set.

Robust Counterpart(RC): The RC of the uncertain LO Problem is the optimization problem

$$\min_x \left\{ \Delta(x) = \sup_{(c,A,b)\in\mathcal{U}} c^T x : Ax \le b, \forall (c,A,b) \in \mathcal{U} \right\}$$

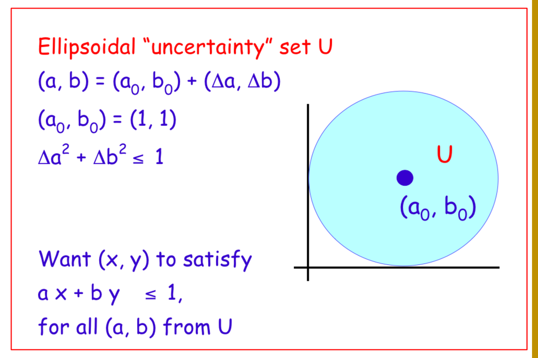
of minimizing the robust value of the objectives over all robust feasible solutions to the problem. Simply the RC of (1) is written in the worst-case realization sense of parameters.

$$\min \; c^T x \quad \text{s.t} \quad Ax \le b. \quad \forall (c,A,b) \in \mathcal{U}$$

3. Uncertain Set and Perturbation

Example: Suppose we have robust optimization problem

$$\min \quad f(x) = 2x_1 + 2x_2$$
$$s.t. \quad a_{11}x_1 + a_{12}x_2 \ge b_1$$
$$a_{21}x_1 + a_{22}x_2 \ge b_2$$

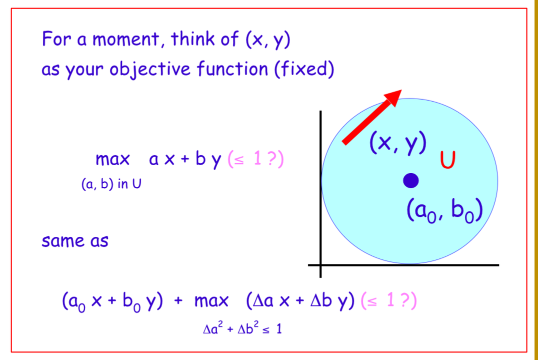


For the nominal value $u = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$ the uncertainty set can be obtained due to perturbation by adding or subtracting any amount to the components from 0 up to and including 0.5 as,

$$\mathcal{U} = \left\{ \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix} + \sum_{l=1}^{6} \xi_l P_l \right\}$$

$$P_1 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix} \quad P_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix} \quad P_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$



and $\xi \in Z = \{\xi = (\xi_1, \dots, \xi_6) \in R^6 \mid -1 \le \xi_i \le 1 \,;\, i = 1, 2, \dots, 6\}$. The set Z is called the perturbation set. Corresponding to each realizations of ξ_i in the interval [-1,1], there is an element in $\mathcal{U}$.

Generally, the perturbation set can be written in the form

$$\mathcal{U} = \left\{ [a; b] = [a^0; b^0] + \sum_{l=1}^{L} \xi_l P_l \,:\, \xi \in Z \right\} \qquad (2)$$

4. Objectives

Being an emerging field, very few works have been developed in the field of robust optimization. Finding the counterpart of a given robust optimization problem and making the RC into a tractable form is a challenging task. The complexity of the RC and the solution techniques depend mostly on the shapes of uncertain sets. The concept of RC and its tractable form in the book [1] and the research paper [2] motivate to work with numerical problems. In this present work we mainly focus on the theoretical improvements and the solutions procedures of numerical problems based on the concept of RC.

5. Solution Methods

The concepts of tractability of RC motivates us to solve certain type of robust LO problems. The RC with interval and ellipsoidal uncertainty have been improved.

General RO Problem: $\max \quad f(x)$

$$s.t. \quad u_{11}(\xi)x_1 + u_{12}(\xi)x_2 + \cdots + u_{1n}(\xi)x_n$$
$$u_{21}(\xi)x_1 + u_{22}(\xi)x_2 + \cdots + u_{2n}(\xi)x_n$$
$$u_{m1}(\xi)x_1 + u_{m2}(\xi)x_2 + \cdots + u_{mn}(\xi)x_n$$

Where each u_{ij} is a function of uncertain parameter ξ from known uncertain set $\mathcal{U}$.

Uncertainty Wise Solution: For a linear RO problem with certain objective function

$$\max \quad f(x) = 4x_1 + 2x_2 + 5x_3$$

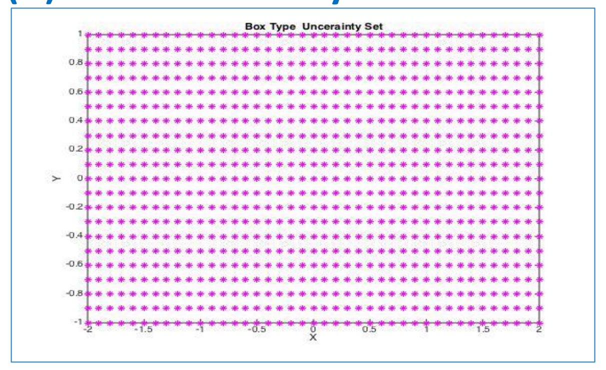
$$s.t. \quad (1 - \xi_1 + 2\xi_2)x_1 + (1 + 2\xi_1 - \xi_2)x_2 + (2 - 2\xi_1 + 3\xi_2)x_3 \le 15 \qquad (3)$$

$$(\xi_1 + \xi_2)x_1 + (2\xi_1 - 3\xi_2)x_2 + (3 - 2\xi_1)x_3 \le 20 \quad \forall x \in \mathcal{U}.$$

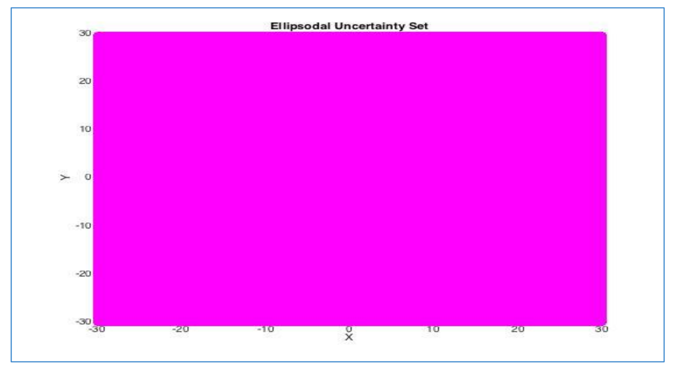
$$\text{where } \mathcal{U} : \begin{cases} \textbf{Box} = \{\xi \mid -a \le \xi_1 \le a \,;\, -b \le \xi_1 \le b : a, b \in N\} \\ \textbf{Ellipse} \;=\; \frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2} = 1 \;: a, b \in N \end{cases}$$

is the uncertainty set and x_1, x_2, x_3 are non-negative variables, we solve the problems and get the optimal solutions.

(A) Box Uncertainty

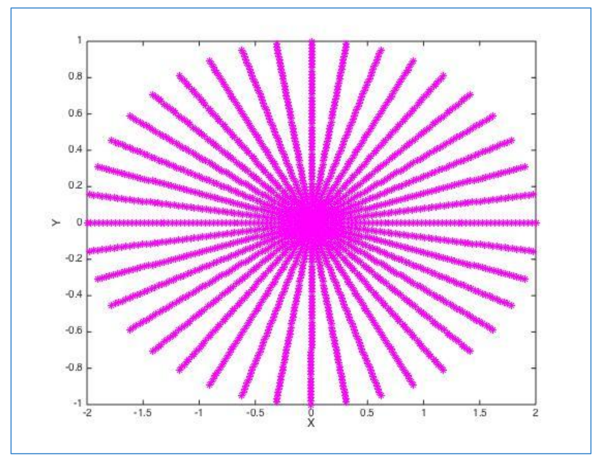


(a) small number of grids

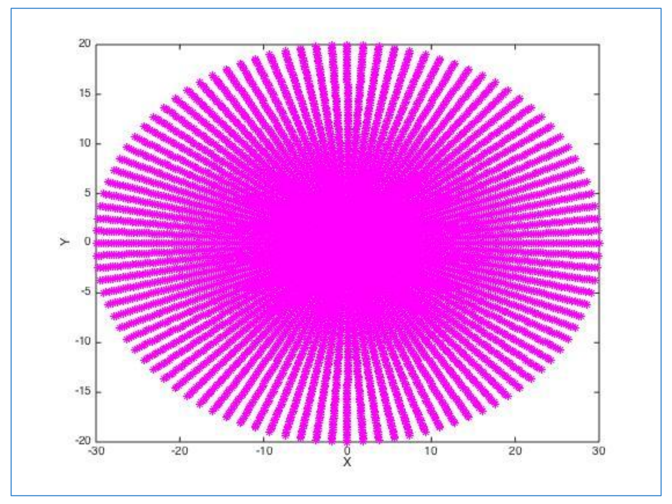


(b) Large number of grids

(B) Ellipsoidal Uncertainty



(a) small number of elliptic grids



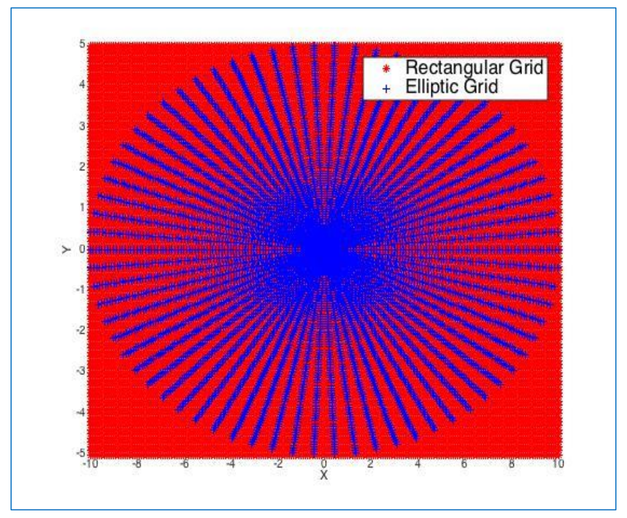
(b) Large number of elliptic grids

6. Results and Discussions

Comparison of The Results: For a particular robust linear maximization problem with certain objective and certain right hand side , we compare the optimal solutions and optimal objective values of (3) under the box and elliptic uncertainty sets. Several observations in the table below help to clear the comparison decision.

Table 1: Comparison of the solutions under box and elliptic uncertainty sets

Unct Reg	OptSol(B)	OptVal	OptSol(E)	OptVal
$a = 1, b = 1$	(5.00, 2.50, 0)	25.000	(4.707, 3.96, 0)	26.758
$a = 2, b = 1$	(4.74, 2.18, 0)	23.333	(4.79, 2.58, 0.121)	24.953
$a = 2, b = 2$	(3.33, 1.67, 0)	16.667	(3.33, 2.71, 0)	18.770
$a = 5, b = 5$	(1.64, 0.80, 0)	8.150	(1.27, 1.20, 0.33)	9.122
$a = 10, b = 5$	(0.01, 1.12, 1.12)	7.882	(0.40, 0.40, 0.89)	8.294
$a = 10, b = 10$	(0.02, 0.61, 0.62)	4.382	(0.59, 0.65, 0.26)	4.954
$a = 20, b = 10$	(0.01, 0.61, 0.62)	4.349	(0.18, 0.62, 0.53	4.591
$a = 20, b = 20$	(0.01, 0.32, 0.33)	2.315	(0.28, 0.34, 0.16)	2.594
$a = 30, b = 20$	(0.01, 0.32,0.32)	2.302	(0.15, 0.33,0.24)	2.473
$a = 30, b = 30$	(0.01, 0.22,0.22)	1.573	(0.18, 0.23, 0.12)	1.758



Simultaneous grid points

Discussion:

- Solution of this RO problem (3) satisfies all the realizations of uncertain parameters values and therefore the solution is robust.
- We briefly compare the optimal solutions and maximum objective values of (3) under box and elliptic uncertainty sets.
- The bigger values of major and minor axes (resp. dimension of box) reduce the maximum value of objective function.
- Maximum objective values do not cross the positivity due to non-negativity conditions of decision variables.
- A figure makes the realization of how much gap is there in the optimum value and how the optimal solutions are fluctuating compared to other uncertainty sets.

Decision: within a certain dimension the ellipsoidal uncertainty set gives better result for a maximization robust linear problem.

Remark: A similar research avers that for a certain dimension box uncertainty set provides the better result for a minimization robust linear problem.

7. Conclusion

- Real-world optimization problems that come from design of physical or engineering systems, often contain parameters whose values can not be measured exactly because of various technical difficulties, or due to incomplete data. Because of the worst case scenario such parameter uncertainties could negatively affect the quality of solutions while trying to achieve a solution as good as possible.
- The proposed robust formulations are theoretically valid only in a neighborhood of the nominal value. Therefore, their solutions will likely be dependent on he quality of parameter estimations and on the shape of the uncertainty sets.

8. References

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